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DYNAMICS OF ASYNCHRONOUS MACHINE WITH A ROTOR OF SQUIRREL-CAGE TYPE

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A well-known method of analytical mechanics consisting of applying ideal constraints is used to construct the equations of motion for a model of an asynchronous machine with a squirrel-cage type rotor, and the dynamics of this machine is studied for the case when the time constant of the mechanical motion is much greater than the time constant of the electrical processes. The ideal character of the constraints is proved and it is shown that from the dynamical points of view, the squirrel-cage type rotor is equivalent to two orthogonal sinusoidal windings only in the case, as Kron postulates [2], when the stator windings have sinusoidally distributed turns.

Although the theory of electric machines with rotating rotors which treats these machines as electromechanical systems has undergone a considerable development, the collectorless machines with solid or squirrel-cage type rotors appear to have fallen outside the scope of this theory [1]. The only relevant theoretical work known to the authors is based on the assumption of Kron [2] that a squirrel-cage type rotor is equivalent to two orthogonal sinusoidal windings. This assumption is however still unproved.

1. Auxilliary model of electric machine. We derive the equations of motion for an asynchronous machine with a squirrel-cage type rotor, using the auxilliary model proposed in [3]. The model consists of two coaxial hollow cylinders of radii a and b with thin conducting walls. The outer cylinder (stator) is fixed and the inner cylinder (rotor) rotates about the common axis O (see Fig. 1). It is assumed that both cylinders are anisotropically conducting, i.e. the stator and the rotor currents flow along the cylinder generatrices and the lengths of the cylinders is such that the end boundary effects can be neglected.

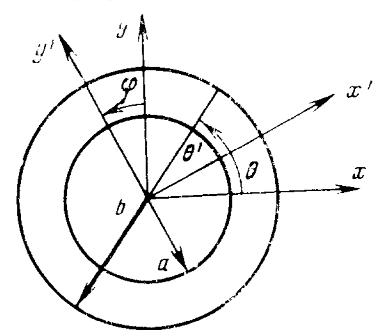

Fig. 1

Let the total current across a section of each cylinder be equal to zero. The rotor and stator current densities j_p and j_c (*) can be expressed in the form of series [4]

$$j_p = \frac{1}{a}\sum_{k=1}^{\infty} \underset{1}{\varkappa_k^{\bullet}} \cos k\theta' + \underset{2}{\varkappa_k^{\bullet}} \sin k\theta', \quad j_c = \frac{1}{b}\sum_{k=1}^{\infty} \underset{3}{\varkappa_k^{\bullet}} \cos k\theta + \underset{4}{\varkappa_k^{\bullet}} \sin k\theta \quad (1.1)$$

*) Editor's note. Symbols p and c used in this paper as subscripts or superscripts, refer to rotor and stator, respectively.

Here the angle θ is taken in the fixed coordinate system and the angle θ' in the rotating coordinate system attached to the rotor. With the currents given in the discrete form shown above, the quantities

$$\varkappa_{\underset{1}{k}},\ \varkappa_{\underset{2}{k}},\ \varkappa_{\underset{3}{k}},\ \varkappa_{\underset{4}{k}}$$

are generalized coordinates. The Lagrangian function of the auxilliary model has the form

$$L_* = \frac{1}{2}\sum_{k=1}^{\infty}\{L_k(\varkappa_{\underset{1}{k}}^{\cdot 2} + \varkappa_{\underset{2}{k}}^{\cdot 2} + \varkappa_{\underset{3}{k}}^{\cdot 2} + \varkappa_{\underset{4}{k}}^{\cdot 2} + 2M[(\varkappa_{\underset{1}{k}}^{\cdot}\cos k\varphi - \tag{1.2}$$

$$\varkappa_{\underset{2}{k}}^{\cdot}\sin k\varphi)\varkappa_{\underset{3}{k}}^{\cdot} + (\varkappa_{\underset{1}{k}}^{\cdot}\sin k\varphi + \varkappa_{\underset{2}{k}}^{\cdot}\cos k\varphi)\varkappa_{\underset{4}{k}}^{\cdot}]\} + \frac{1}{2}I\varphi^{\cdot 2}$$

where I is the moment of inertia of the rotor, L_k is the coefficient of self-inductance and M_k is the coefficient of mutual inductance; we have

$$L_k = 2\left(\frac{\pi}{c}\right)^2\frac{\mu}{k}, \qquad M_k = 2\left(\frac{\pi}{c}\right)^2\left(\frac{a}{b}\right)^k \qquad (k = 1, 2, \ldots)$$

The generalized forces are determined from the expression for the virtual work

$$\delta A = \sum_{k=1}^{\infty}\Big[\sum_{i=1}^{2}(E_{ik} - R_1\varkappa_{\underset{i}{k}}^{\cdot})\,\delta\varkappa_{\underset{i}{k}} + \sum_{j=3}^{4}(E_{jk} - R_2\varkappa_{\underset{j}{k}}^{\cdot})\,\delta\varkappa_{\underset{j}{k}}\Big] + \tag{1.3}$$

$$K(\varphi^{\cdot})\,\delta\varphi = \sum_{l=1}^{\infty}Q_l\delta q_l$$

The rotor and stator resistances R_{rk}^{p} and R_{rk}^{c} are computed from the formulas

$$R_{rk}^{p} = \frac{1}{a}\int_0^{2\pi}\frac{1}{\sigma_1}\begin{matrix}\sin\\ \cos\end{matrix}\, r\theta' \begin{matrix}\sin\\ \cos\end{matrix}\, k\theta'\,d\theta', \qquad R_{rk}^{c} = \frac{1}{b}\int_0^{2\pi}\frac{1}{\sigma_2}\begin{matrix}\sin\\ \cos\end{matrix}\, r\theta \begin{matrix}\sin\\ \cos\end{matrix}\, k\theta\,d\theta$$

and the equations of motion of the auxilliary model are written in the form ot Lagrange-Maxwell equations

$$\frac{d}{dt}\frac{\partial L_*}{\partial q_k^{\cdot}} - \frac{\partial L_*}{\partial q_k} = Q_k \qquad (k = 1, 2, \ldots) \tag{1.4}$$

which are constructed using the expressions (1.2) and (1.3).

2. Asynchronous machine with sinusoidal windings on the stator. We shall now consider a model of an asynchronous machine with the squirrel-cage type rotor, which is obtained by imposing ideal constraints on the auxilliary model. The realization of these constraints which restrict the currents in the auxilliary model presupposes the presence of certain additional emf (electromotive force(s)) which, by analogy with the classical mechanics, shall be called the generalized constraint reaction forces. We shall see further that in the present case the constraints are ideal, because certain currents $q_i^{\cdot}$ describing the state of the system, vanish. The virtual work done by the additional emf (constraint reaction forces) also vanishes, since the relations $\delta q_i = 0$ hold on the segments at which these emf are applied (the situation is analogous to the case of a solid rolling without slipping). It will be shown that this is precisely what happens when we pass from the auxilliary model to the asynchronous model of the electric machine.

For the sake of definiteness we shall consider two models of an asynchronous machine with a squirrel-cage type rotor. In the first model the stator has the windings with sinusoidally distributed turns, and in the second model the turns of the windings are distributed uniformly.

We begin by considering the simplest case when the stator has two windings with sinusoidally distributed turns. As we know, to generate a rotating magnetic field it is sufficient to place the magnetic axes of these windings at right angle to each other and apply to them the emf shifted in phase by $\pi / 2$. Let the turns of the first winding be distributed along the circumference of the stator according to the law $\varepsilon \mid \cos\theta \mid$, and the turns of the second winding — according to $\varepsilon \mid \sin\theta \mid$. If a unit cross section area of each winding contains m conductors carrying the currents $q_1^{\bullet}$ in the first winding and $q_2^{\bullet}$ in the second winding, then the combined current density in the stator windings is equal to

$$j_c(\theta) = \varepsilon m (q_1^{\bullet} \cos\theta + q_2^{\bullet} \sin\theta)$$

Comparing this expression with (1.1) we find, that in the present case the model of an asynchronous electric machine can be obtained from the auxilliary model by imposing on it the following constraints:

$$\underset{3}{\varkappa_k^{\bullet}} = \underset{4}{\varkappa_k^{\bullet}} = 0 \qquad (k = 2, 3, \ldots)$$

To realize these constraints, it is sufficient to apply additional emf across the circuits of the auxilliary model through which the currents

$$\underset{3}{\varkappa_k^{\bullet}}, \quad \underset{4}{\varkappa_k^{\bullet}} \qquad (k = 2, 3, \ldots)$$

are flowing, the emf being of such magnitude and polarity that the resulting current through these circuits becomes zero. We have said above that the virtual work done by the additional emf is equal to zero, therefore the constraints imposed are ideal.

The Lagrangian function (1.2) now becomes

$$L_* = \frac{1}{2} L (q_1^{\bullet 2} + q_2^{\bullet 2}) + M (\underset{1}{\varkappa_1^{\bullet}} \cos\varphi - \underset{2}{\varkappa_1^{\bullet}} \sin\varphi) q_1^{\bullet} +$$

$$(\underset{1}{\varkappa_1^{\bullet}} \sin\varphi + \underset{2}{\varkappa_1^{\bullet}} \cos\varphi) q_2^{\bullet}] + \frac{1}{2} \sum_{k=1}^{\infty} L_k (\underset{1}{\varkappa_k^{\bullet 2}} + \underset{2}{\varkappa_k^{\bullet 2}}) + \frac{1}{2} I \varphi^{\bullet 2}$$

where the following notation is used:

$$\underset{3}{\varkappa_1^{\bullet}} = \beta q_1^{\bullet}, \quad \underset{4}{\varkappa_1^{\bullet}} = \beta q_2^{\bullet}, \quad L = L_1 \beta^2 = 2\mu\beta^2 (\pi / c)^2$$

$$M = \beta M_1 = 2\alpha\mu (\pi / c)^2, \qquad \alpha = ma\varepsilon, \quad \beta = mb\varepsilon$$

Let us denote the ohmic resistance of each stator winding by R and introduce the generalized forces by means of the following expressions:

$$Q_{q_1} = (E + e) \cos\omega_1 t - R q_1^{\bullet}, \qquad Q_{q_2} = E \sin(\omega_1 + \psi) - R q_2^{\bullet}$$

$$Q_{\underset{1}{\varkappa_k}} = - R \underset{1}{\varkappa_k^{\bullet}}, \quad Q_{\underset{2}{\varkappa_k}} = - R \underset{2}{\varkappa_k^{\bullet}} \quad (k = 1, 2, \ldots), \quad Q_{\varphi} = K(\varphi^{\bullet})$$

where we assume, for the sake of generality, that the emf applied across the stator windings differ from each other in amplitude by the quantity e, and in phase by ψ.

The equations of motion of the model of an asynchronous machine under consideration, have the form

$$L q_1^{\bullet\bullet} + M \frac{d}{dt} (\underset{1}{\varkappa_1^{\bullet}} \cos\varphi - \underset{2}{\varkappa_1^{\bullet}} \sin\varphi) + R q_1^{\bullet} = (E + e) \cos\omega_1 t \quad (2.1)$$

$$I q_2^{\bullet\bullet} + M \frac{d}{dt} (\underset{1}{\varkappa_1^{\bullet}} \sin\varphi + \underset{2}{\varkappa_1^{\bullet}} \cos\varphi) + R q_2^{\bullet} = E \sin(\omega_1 t + \psi)$$

$$L_1 \underset{1}{\varkappa_1}^{\cdot\cdot} + M \frac{d}{dt}(q_1^{\cdot} \cos\varphi + q_2^{\cdot} \sin\varphi) + R_1 \underset{2}{\varkappa_1}^{\cdot} = 0$$

$$L_1 \underset{2}{\varkappa_1}^{\cdot\cdot} + M \frac{d}{dt}(- q_1^{\cdot} \sin\varphi + q_2^{\cdot} \cos\varphi) + R_1 \underset{2}{\varkappa_1}^{\cdot} = 0$$

$$L_k \underset{1}{\varkappa_k}^{\cdot\cdot} + R_1 \underset{1}{\varkappa_k}^{\cdot} = 0, \quad L_k \underset{2}{\varkappa_k}^{\cdot\cdot} + R_1 \underset{2}{\varkappa_k}^{\cdot} = 0 \quad (k = 2, 3, \ldots)$$

$$I\varphi^{\cdot\cdot} + M\,[q_1^{\cdot}(\underset{1}{\varkappa_1}^{\cdot} \sin\varphi + \underset{2}{\varkappa_1}^{\cdot} \cos\varphi) - q_2^{\cdot}(\underset{1}{\varkappa_1}^{\cdot} \cos\varphi - \underset{2}{\varkappa_1}^{\cdot} \sin\varphi)] = K(\varphi^{\cdot})$$

Since the variables

$$\underset{1,\,2}{\varkappa_k} \quad (k = 2,\ 3,\ 4, \ldots)$$

have become separated and decay with time, the equations for these variables can be omitted.

In the dimensionless complex variables

$$\zeta = \frac{\omega_1 L}{E}(q_1^{\cdot} + q_2^{\cdot}), \quad \zeta_1 = \frac{\omega_1 L}{E}[\underset{1}{\varkappa_1} \cos\varphi - \underset{2}{\varkappa_1} \sin\varphi + i\,(\underset{1}{\varkappa_1} \sin\varphi + \underset{2}{\varkappa_1} \cos\varphi)]$$

the dynamics of the model under consideration is described by the following system of differential equations:

$$\frac{d\zeta}{d\tau} + \rho\zeta + \mu\frac{d\zeta_1}{d\tau} = e^{i\tau} + \sigma e^{-i\tau} \tag{2.2}$$

$$\frac{d\zeta_1}{d\tau} + \rho_1\zeta_1 - i\xi\zeta_1 + \mu_1\frac{d\zeta}{d\tau} - i\mu_1\xi\zeta = 0$$

$$\frac{d\xi}{d\tau} + M_0 \operatorname{Im}(\zeta^*\zeta_1) = K_0(\xi)$$

$$[\tau = \omega_1 t,\ \xi = \varphi^{\cdot}\omega_1^{-1},\ \sigma = 2E^{-1}(e + i\psi E)]$$

where we utilize the inequalities $e \ll E$ and $\psi \ll 1$. The dimensionless parameters in (2.2) are determined by the expressions

$$\mu = \frac{M}{L}, \quad \mu_1 = \frac{M}{L_1}, \quad \rho = \frac{R}{\omega_1 L}, \quad \rho_1 = \frac{R_1}{\omega_1 L_1},$$

$$M_0 = \frac{ME^2}{IL\omega_1^3}, \quad K_0 = \frac{K}{I\omega_1^2}$$

We note that Eqs. (2.1) readily yield the equations of a single-phase asynchronous machine with a squirrel-case type rotor. These equations have the form

$$Lq_1^{\cdot\cdot} + Rq_1^{\cdot} + M\pi_1^{\cdot\cdot} = E\cos\omega_1 t, \quad L_1\pi_1^{\cdot\cdot} + L_1\varphi^{\cdot}\pi_2^{\cdot} + Mq_1^{\cdot\cdot} + R_1\pi_1^{\cdot} = 0$$

$$I\varphi^{\cdot\cdot} + Mq_1^{\cdot}\pi_2^{\cdot} = K(\varphi^{\cdot}), \quad L_1\pi_2^{\cdot\cdot} - L_1\varphi^{\cdot}\pi_1^{\cdot} - Mq_1^{\cdot}\varphi^{\cdot} + R_1\pi_2^{\cdot} = 0$$

$$(\pi_1^{\cdot} = \underset{1}{\varkappa_1}^{\cdot} \cos\varphi - \underset{2}{\varkappa_1}^{\cdot} \sin\varphi,\ \pi_2^{\cdot} = \underset{1}{\varkappa_1}^{\cdot} \sin\varphi + \underset{2}{\varkappa_1}^{\cdot} \cos\varphi)$$

Let us now consider a model of an asynchronous machine with three sinusoidal windings of the stator; the magnetic axes of these windings are placed at 120° to each other. If m is the number of quasi-linear conductors per unit cross section area of the stator winding and ε is the maximum thickness of a single winding, then the surface current density on the stator j_c is given by

$$j_c = m\varepsilon\left[q_1^{\cdot}\cos\theta + q_2^{\cdot}\cos\left(\theta - \frac{2\pi}{3}\right) + q_3^{\cdot}\cos\left(\theta + \frac{2\pi}{3}\right)\right]$$

where $q_1^{\cdot}$, $q_2^{\cdot}$ and $q_3^{\cdot}$ denote the currents in the corresponding windings. Let us replace the currents $q_2^{\cdot}$ and $q_3^{\cdot}$ by the currents $q_x^{\cdot}$ and $q_y^{\cdot}$ using the following relations:

$$q_2^{\cdot} = q_1^{\cdot} - q_x^{\cdot} + 1/\sqrt{3}\, q_y^{\cdot}, \qquad q_3^{\cdot} = q_1^{\cdot} - q_x^{\cdot} - 1/\sqrt{3}\, q_y^{\cdot}$$

Then the previous expression becomes

$$j_c = m\varepsilon\,(q_x^{\cdot}\cos\theta + q_y^{\cdot}\sin\theta)$$

On the basis of (1.1) we find

$$\underset{3}{\varkappa_1^{\cdot}} = 3q_x^{\cdot}, \quad \underset{4}{\varkappa_1^{\cdot}} = 3q_y^{\cdot}, \quad \underset{3}{\varkappa_k^{\cdot}} = \underset{4}{\varkappa_k^{\cdot}} = 0 \quad (k = 2, 3, \ldots)$$

which implies that the constraints in the present case are ideal. The Lagrangian function has the form

$$L_* = \frac{1}{2}L(q_x^{\cdot 2} + q_y^{\cdot 2}) + M[(\underset{1}{\varkappa_1^{\cdot}}\cos\varphi - \underset{2}{\varkappa_1^{\cdot}}\sin\varphi)\,q_x^{\cdot} + (\underset{1}{\varkappa_1^{\cdot}}\sin\varphi + \underset{2}{\varkappa_1^{\cdot}}\cos\varphi)\,q_y^{\cdot}] + \frac{1}{2}\sum_{k=1}^{\infty} L_k(\underset{1}{\varkappa_k^{\cdot 2}} + \underset{2}{\varkappa_k^{\cdot 2}}) + \frac{1}{2}I\varphi^{\cdot 2}$$

To find the generalized forces we construct an expression for the virtual work

$$\delta A = (E\cos\omega_1 t - Rq_1^{\cdot})\,\delta q_1 + \left[E\cos\left(\omega_1 t - \frac{2\pi}{3}\right) - Rq_2^{\cdot}\right]\delta q_2 + \left[E\cos\left(\omega_1 t + \frac{2\pi}{3}\right) - Rq_3^{\cdot}\right]\delta q_3 + K(\varphi^{\cdot})\,\delta\varphi$$

hence

$$Q_{q_1} = R(2q_x^{\cdot} - 3q_1^{\cdot}), \qquad Q_y = K(\varphi^{\cdot})$$
$$Q_{q_2} = E\cos\omega_1 t + 2R(q_1^{\cdot} - q_2^{\cdot}), \quad Q_{q_y} = E\sin\omega_1 t - {}^2/_3 Rq_y^{\cdot}$$

The function L_* is independent of the coordinate q_1 and of its time derivative, therefore the Lagrange equation (1.4) for this coordinate degenerates to the relation $q_1^{\cdot} = {}^2/_3 q_x^{\cdot}$, from which we can find q_1 using the known function $q_x^{\cdot}$.

The equation of motion for the model of an asynchronous machine become analogous to (2.1) when $e = \psi = 0$, and q_1, q_2 are replaced by q_x, q_y, and R by ${}^2/_3 R$. Consequently the dynamics of the asynchronous machine with three sinusoidal windings on the stator does not differ from the dynamics of the asynchronous machine with two sinusoidal windings. This result can be extended to a stator with an arbitrary number of sinusoidally distributed windings. We can therefore conclude that, when a stator has sinusoidal windings and the corresponding emf necessary to generate a rotating magnetic field are present, then the asynchronous electric machine with the squirrel-cage type rotor is dynamically equivalent to a model the rotor of which has only two closed sinusoidal windings with orthogonal magnetic axes. Thus the Kron's hypothesis [2] is confirmed for the case of the asynchronous electric machine with sinusoidal windings on the stator.

3. Asynchronous machine with a uniform winding on the stator. Let the stator of an asynchronous machine with a squirrel-cage type rotor contain two orthogonal windings with turns uniformly distributed along the circumference. The expression for the current density on the stator can be written in the form

$$j_c = m\varepsilon\,[q_1^{\cdot} f_1(\theta) + q_2^{\cdot} f_2(\theta)]$$

where ε is the thickness of a single winding, m is the number of conductors per unit cross section area of the stator winding and the functions $f_1(\theta)$ and $f_2(\theta)$ are of the form shown on Fig. 2. Then

$$f_1(\theta) = \frac{4}{\pi}\sum_{k=1,3}^{\infty} k^{-1}(-1)^{(k-1)/2}\cos k\theta, \qquad f_2(\theta) = \frac{4}{\pi}\sum_{k=1,3}^{\infty} k^{-1}\sin k\theta$$

Equating the expression for j_c with (1.1) we obtain

$$\underset{3}{\varkappa_k^{\cdot}} = 4\beta(-1)^{(k-1)/2} q_1^{\cdot}, \qquad \underset{4}{\varkappa_k^{\cdot}} = \frac{4\beta}{k\pi} q_2^{\cdot} \quad (k = 1, 3, \ldots) \tag{3.1}$$

$$\underset{3}{\dot{\varkappa}_{2n}} = \underset{4}{\dot{\varkappa}_{2n}} = 0 \qquad (n = 1, 2, \ldots)$$

The last relations in (3.1) represent, in accordance with the previous statements, the equations of the ideal constraints. We shall show that the remaining expressions in (3.1) also represent the equations of ideal constraints. Let us e. g. consider the enumerable set

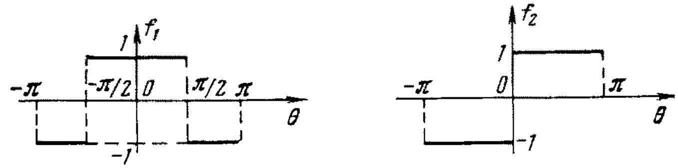

Fig. 2

of the second group of equations in (3.1). They reflect the fact that the currents

$$\underset{4}{\varkappa_k^{\cdot}} \quad (k = 1, 3, \ldots)\cdot$$

are interdependent, namey

$$\underset{4}{\varkappa_3^{\cdot}} = {}^1/_3 \underset{4}{\varkappa_1^{\cdot}}, \qquad \underset{4}{\varkappa_5^{\cdot}} = {}^1/_5 \underset{4}{\varkappa_1^{\cdot}}, \ldots \tag{3.2}$$

Moreover, the relation

$$\underset{4}{\varkappa_1^{\cdot}} = (4\beta/\pi)\, q_2^{\cdot}$$

can be considered as another expression for the current $\underset{4}{\varkappa_1^{\cdot}}$. Before the imposition of the constraints (3.2), the currents

$$\underset{4}{\varkappa_k^{\cdot}} \quad (k = 1, 3, \ldots)$$

in the stator circuits were governed, in accordance with (1.4), by the equations (Fig. 3a)

$$L_k \underset{4}{\varkappa_k^{\cdot\cdot}} + R_2 \underset{4}{\varkappa_k^{\cdot}} = \underset{4}{E_k} + \underset{4}{E_k^{\,i}} \qquad (k = 1, 3, \ldots)$$

$$\underset{4}{E_k^{\,i}} = -M_k \frac{d}{dt}\left(\underset{1}{\varkappa_k^{\cdot}}\sin k\varphi + \underset{2}{\varkappa_k^{\cdot}}\cos k\varphi\right)$$

Let us see how these circuits can be fitted into a new diagram, which can be obtained from the diagram Fig. 3a by replacing its part outlined by a dashed line, by that given in Fig. 3b. Here $R_1, R_3, \ldots$ represent the additional controlled resistances and $E_1, E_3, \ldots$ are the additional controlled emf. We can choose $R_1(t), R_3(t), \ldots E_1(t), E_3(t), \ldots$ so that the relations (3.2) hold and the virtual work done by the additional emf $\underset{4}{E}_1(t)$, $\underset{4}{E}_3(t)$, . . . applied in order to realize the constraints (3.2) is zero. In fact, the condition for the constraints to be ideal is, that the relations

$$i_1 = \underset{4}{\varkappa}_3^{\cdot}, \qquad i_3 = i_1 + \underset{4}{\varkappa}_5^{\cdot}, \ \ldots \tag{3.3}$$

where i_1, i_3, . . . are the additional circuit currents, hold. The condition (3.3) means that the total current on the segments A_1B_1, A_2B_2, . . . is equal to zero. Let us write

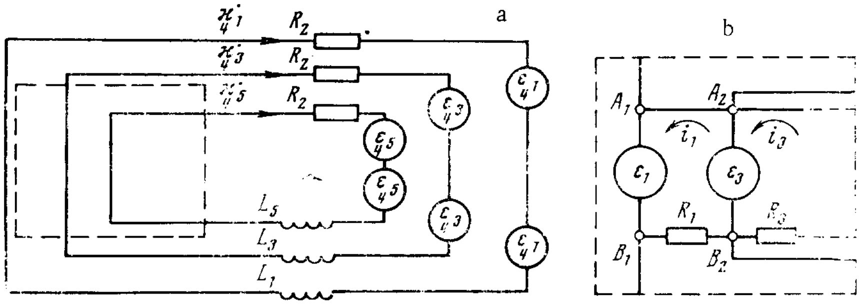

Fig. 3

the Kirchhoff equations for the new diagram

$$\begin{array}{ll} L_1\underset{4}{\varkappa}_1^{\cdot\cdot} + R_2\underset{4}{\varkappa}_1^{\cdot} = \underset{4}{E}_1 + \underset{4}{E}_1^{\,i}, & R_1 i_1 = E_1 - E_3 \\ L_3\underset{4}{\varkappa}_3^{\cdot\cdot} + R_2\underset{4}{\varkappa}_3^{\cdot} = \underset{4}{E}_3 + \underset{4}{E}_3^{\,i} - E_1, & R_3 i_3 = E_3 - E_5 \\ L_5\underset{4}{\varkappa}_5^{\cdot\cdot} + R_2\underset{4}{\varkappa}_5^{\cdot} = \underset{4}{E}_5 + \underset{4}{E}_5^{\,i} - E_3, & R_5 i_5 = E_5 - E_7 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{array}$$

From the above equations it follows that if E_1, E_3, . . . are defined by the expressions

$$\begin{aligned} E_1 &= \underset{4}{E}_3 + \underset{4}{E}_3^{\,i} - {}^1/_3 L_3\underset{4}{\varkappa}_1^{\cdot\cdot} - {}^1/_3 R_2\underset{4}{\varkappa}_1^{\cdot} \\ E_3 &= \underset{4}{E}_5 + \underset{4}{E}_5^{\,i} - {}^1/_5 L_5\underset{4}{\varkappa}_1^{\cdot\cdot} - {}^1/_5 R_2\underset{4}{\varkappa}_1^{\cdot} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3.4}$$

then the relations (3.2) hold; for (3.3) to be held it is sufficient to define R_1, R_3, . . . in the form

$$R_1 = \frac{3(E_1 - E_3)}{\underset{4}{\varkappa}_1^{\cdot}}, \qquad R_3 = \frac{15(E_3 - E_5)}{8\underset{4}{\varkappa}_1^{\cdot}}, \ \ldots$$

where the quantities E_1, E_3, . . . are replaced by their expressions given in (3.4).

An analogous proof can be constructed for the other group of equations in (3.1). Using the relations (3.1) and introducing the notation

$$L_0 = \left(\frac{4\beta}{\pi}\right)^2 \sum_{k=1,\,3}^{\infty} \frac{L_k}{k^2}, \qquad M_k^{\circ} = \frac{4\beta M_k}{k\pi} \quad (k = 1, 3, \ldots)$$

we obtain from (1.2) the Lagrangian function, which in the case under consideration, is

$$L_* = \frac{1}{2}\Big[L_0(q_1^{\cdot 2} + q_2^{\cdot 2}) + \sum_{n=1}^{\infty} L_n(\underset{1}{\varkappa}_n^{\cdot 2} + \underset{2}{\varkappa}_n^{\cdot 2})\Big] + \frac{1}{2} I\varphi^{\cdot 2} +$$

$$\sum_{k=1,3}^{\infty} M_k^{\circ} [q_1^{\cdot} (\underset{1}{\varkappa_k}^{\cdot} \cos k\varphi - \underset{2}{\varkappa_k}^{\cdot} \sin k\varphi) + q_2^{\cdot} (\underset{1}{\varkappa_k}^{\cdot} \sin k\varphi + \underset{2}{\varkappa_k}^{\cdot} \cos k\varphi)]$$

From the expression for the virtual work we find the generalized forces

$$Q_{q_1} = (E + e) \cos \omega_1 t - R_0 q_1^{\cdot}, \qquad Q_{q_2} = E \sin (\omega_1 t + \psi) - R_0 q_2^{\cdot}$$

$$Q_{\underset{1}{\varkappa_n}} = - R_1 \underset{1}{\varkappa_n}^{\cdot}, \quad Q_{\underset{2}{\varkappa_n}} = - R_1 \underset{1}{\varkappa_n}^{\cdot} \quad (n = 1, 2, \ldots), \quad Q_{\varphi} = K(\varphi^{\cdot})$$

Now using (1.4) we can easily construct the equations of motion for the asynchronous machine with a uniform winding on the stator.

In the dimensionless variables

$$\zeta_k = \frac{\omega_1 L_0}{E} [\underset{1}{\varkappa_k}^{\cdot} \cos k\varphi - \underset{2}{\varkappa_k}^{\cdot} \sin k\varphi + i(-1)^{(k-1)/2} \times \tag{3.5}$$
$$(\underset{1}{\varkappa_k}^{\cdot} \sin k\varphi - \underset{2}{\varkappa_k}^{\cdot} \cos k\varphi)]$$

$$\zeta = \frac{\omega_1 L_0}{E} (q_1^{\cdot} + i q_2^{\cdot}), \quad \tau = \omega_1 t, \quad \xi = \frac{\varphi^{\cdot}}{\omega_1}, \quad \lambda_k = \frac{M_k^{\cdot}}{L_0}$$

$$\mu_k = \frac{M_k^{\circ}}{L_k}, \quad \rho = \frac{R_0}{\omega_1 L_0}, \quad \rho_k = \frac{R_1}{\omega_1 L_k}$$

$$\gamma = \frac{E^2}{J \omega_1^3 L_0}, \quad K_0 = \frac{K}{J \omega_1^2}, \quad \sigma = \frac{1}{2E} (e + i\psi E)$$

the dynamics of the asynchronous machine in question is described by the following system of differential equations

$$\frac{d\zeta}{d\tau} + \rho\zeta + \sum_{k=1,3}^{\infty} (-1)^{(k-1)/2} \lambda_k \frac{d\zeta_k}{d\tau} = e^{i\tau} + \sigma e^{-i\tau} \tag{3.6}$$

$$\frac{d\zeta_k}{d\tau} + \rho_k \zeta_k + (-1)^{(k-1)/2} \mu_k \frac{d\zeta}{d\tau} - ik\xi (\zeta_k + \mu_k \zeta) = 0$$

$$\frac{d\xi}{d\tau} + \gamma \sum_{k=1,3}^{\infty} k\lambda_k \operatorname{Im} \{\zeta^* \zeta_k\} = K_0(\xi)$$

The form of the above equations indicates that in the case of an asynchronous machine with a uniform winding on the stator, the squirrel-cage type rotor is not equivalent to two windings with orthogonal magnetic axes, i.e. the Kron's assumption [2] does not hold.

Since setting $k = 1$ in the system (3.6) converts it to (2.2) which describes the dynamics of an asynchronous machine with sinusoidal windings on the stator, we study the dynamics of both models at the same time.

4. Investigation of the dynamics of the asynchronous machine. Let us consider two models of the asynchronous machine with a squirrel-cage type rotor described by the equations (2.2) and (3.6), the latter referring to the case $\sigma = 0$, i.e. to the case of the stator windings under a symmetrical load. To simplify the investigation, we shall use the following change of variables:

$$\zeta = u e^{i\tau}, \quad \zeta_k = v_k e^{i\tau} \qquad (u = u' + iu'', \ v_k = v_k' + i v_k'')$$

which enables us to consider, instead of the nonautonomous system of differential equations (3.6), the autonomous system

$$\frac{du}{d\tau} + (\rho + i)u + \sum_{k=1,3}^{\infty} (-1)^{(k-1)/2}\lambda_k\left(\frac{dv_k}{d\tau} + iv_k\right) = 1 \tag{4.1}$$

$$\frac{dv_k}{d\tau} + [\rho_k + i(1 - k\xi)]\,v_k + (-1)^{(k-1)/2}\mu_k\frac{du}{d\tau} + i\mu_k[(-1)^{(k-1)/2} - k\xi]\,u = 0$$

$$\frac{d\xi}{d\tau} + \gamma\sum_{k=1,3}^{\infty} k\lambda_k(u'v_k'' - u''v_k') = K_0(\xi) \quad (k = 1, 3, \ldots)$$

The state of equilibrium of the system (4.1) corresponding to the steady-state mode of the asynchronous machine under consideration is determined by the values of the variables

$$u' = a(a^2 + b^2)^{-1}, \qquad u'' = -b(a^2 + b^2)^{-1} \tag{4.2}$$

$$v_k' = S_k[\rho_k u'' - (1 - k\xi)u'], \quad v_k'' = -S_k[\rho_k u' + (1 - k\xi)u'']$$

where the following notation is used:

$$a = \rho + \sum_{k=1,3}^{\infty} \lambda_k\rho_k(-1)^{(k-1)/2}S_k, \quad b = 1 - \sum_{k=1,3}^{\infty}(-1)^{(k-1)/2}\times \lambda_k(1 - k\xi)\,S_k \tag{4.3}$$

$$S_k = \mu_k[(-1)^{(k-1)/2} - k\xi][\rho_k^2 + (1 - k\xi)^2]^{-1} \quad (k = 1, 3, \ldots)$$

The quantity ξ is a root of the equation

$$\gamma(a^2 + b^2)^{-1}\sum_{k=1,3}^{\infty} k\lambda_k\rho_k S_k + K_0(\xi) = 0 \tag{4.4}$$

Let us consider the case when the time constant of the mechanical motion of the rotor is much larger than the build-up time of the electrical processes. Let $\gamma = \varepsilon \ll 1$ and $K_0(\xi) = \varepsilon\, T(\xi)$, where $T(\xi) = \gamma^{-1}K_0(\xi)$ is a finite quantity. We introduce the "slow" time $\tau_0 = \varepsilon\tau$ and write the system (4.1) in the form

$$\varepsilon\frac{du}{d\tau_0} + (\rho + i)u + \sum_{k=1,3}^{\infty}(-1)^{(k-1)/2}\left(\varepsilon\frac{dv_k}{d\tau_0} + iv_k\right) = 1 \tag{4.5}$$

$$\varepsilon\frac{dv_k}{d\tau_0} + [\rho_k + i(1 - k\xi)]\,v_k + (-1)^{(k-1)/2}\mu_k\varepsilon\frac{du}{d\tau_0} + i\mu_k[(-1)^{(k-1)/2} - k\xi]\,u = 0$$

$$\frac{d\xi}{d\tau_0} + \sum_{k=1,3}^{\infty} k\lambda_k(u'v_k'' - u''v_k') = T(\xi) \qquad (k = 1, 3, \ldots) \tag{4.6}$$

When $\varepsilon \to 0$, the motion of the system consists of rapid motions with respect to the variables u and v_k and a slow motion with respect to ξ. The rapid motions are described by the system (4.5) in which the derivative is accompanied by a small parameter. The rapid motions can be studied by setting $\xi = \text{const}$ in Eqs. (4.5), whereupon they become linear equations. The stability of equilibrium which is attained by the variables u and v_k $(k = 1, 3, \ldots)$, in a discontinuous manner (as $\varepsilon \to 0$) is determined by the roots p of the characteristic equation

$$p+\rho+i-A(p+i)\sum_{k=1,3}^{\infty}\frac{\alpha^{2k}}{k^3}\frac{p+i-i(-1)^{(k-1)/2}k\xi}{p+i+\rho_k-ik\xi}=0 \qquad (4.7)$$

$$A=\Big(\sum_{k=1,3}^{\infty}k^{-3}\Big)^{-1}, \qquad \alpha=ab^{-1}<1$$

Setting $z=p+i$, we introduce the parameter w and write (4.7) in the form

$$w=A\frac{z}{z+\rho}\sum_{k=1,3}^{\infty}\frac{\alpha^{2k}}{k^3}\frac{z-i(-1)^{(k-1)/2}k\xi}{z+\rho_k-ik\xi} \qquad (4.8)$$

Thus, instead of (4.7) we shall investigate a family of characteristic equations (4.8) depending on the parameter $w\,(0\leqslant w\leqslant\infty)$. The point $w=1$ corresponds to the initial characteristic equation (4.7). From (4.8) we see that if $\rho<1$ and all $\rho_k>0$ $(k=1, 3, \ldots)$, then the point $w=\infty$ belongs to the domain of stability. The curve $w=w(i\omega)$ which is obtained from (4.8) by making the substitution $p=i\omega$ and varying ω from $-\infty$ to $+\infty$, determines the boundary of the D-partition [5] on the complex plane and, consequently, the boundary of the domain of stability with respect to the parameter w. We shall show that this boundary passes to the left of the point $w=1$. In fact, the equation for $w=w(i\omega)$ can be written in the form

$$w(i\omega)=AP_0(i\omega)\sum_{k=1,3}^{\infty}\frac{\alpha^{2k}}{k^3}P_k(i\omega)$$

where $P_0(i\omega)=0$ and $P_k(i\omega)=0$ $(k=1, 3, \ldots)$ represent the equations of circles passing through the coordinate origin and intersecting the abscissa at the point $w=1$, when $w=\pm\infty$. From this we find, taking into account the inequality $\alpha<1$, that

$$w(\pm i\omega)=A\sum_{k=1,3}^{\infty}\alpha^{2k}k^{-1}<A\sum_{k=1,3}^{\infty}k^{-3}=\frac{A}{A}=1$$

Thus the D-partition curve passes to the left of the point $w=1$ and both points $w=1$ and $w=\infty$ belong to the domain of stability provided that the following inequalities hold:

$$\rho>0, \qquad \rho_k>0 \quad (k=1, 3, \ldots)$$

When the ohmic resistance is present in the rotor and in the stator windings, the above inequalities always hold, consequently, the states of equilibrium for the rapid variables u and v_k $(k=1, 3, \ldots)$ are stable. The values of the variables u and v_k in the state of equilibrium are determined from the expressions (4.2). Substituting these expressions into (4.6), we obtain the following equation describing the dynamics of the asynchronous machine in the present case:

$$d\xi/d\tau_0=M_*(\xi)+T(\xi) \qquad (4.9)$$

where the electromechanical moment $M_*(\xi)$ in the notation (4.3) is determined by the expression

$$M_*(\xi)=(a^2+b^2)^{-1}\sum_{k=1,3}^{\infty}k\lambda_k\rho_kS_k \qquad (4.10)$$

The study of the dynamics of an asynchronous machine reduces in the present case to the process of partitioning the phase line ξ into trajectories, i.e. to determining the states of equilibrium on this straight line and defining their stability. The problem is solved by constructing a graph of the curve $f(\xi) = M_*(\xi) + T(\xi)$. The points of intersection of this curve with the abscissa determine the states of equilibrium of (4.9) corresponding to the steady-state modes of the asynchronous machine, and the sign of the derivative $f'(\xi_i)$ defines the stability of the mode in question.

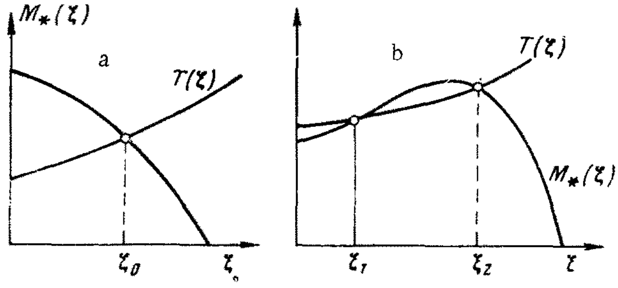

Fig. 4

Instead of the curve $f(\xi)$ we can construct the graphs of the curves $M_* = M_*(\xi)$ and $T = T(\xi)$ and then investigate the character of their points of intersection. As an example, we present a graph (see Fig. 4) showing the dependence of the electromechanical moment $M_*(\xi)$ on the angular velocity ξ of rotation of the rotor for an asynchronous machine with sinusoidal windings on the stator. The expression for this moment is obtained from (4.10) for $k = 1$ and has the form

$$M_*(\xi) = \frac{\lambda_1\mu_1\rho_1 s}{\rho_1^2(1+\rho^2) + 2\lambda_1\mu_1\rho\rho_1 s + [\rho^2 + (1-\lambda_1\mu_1)^2]\, s^2} \qquad (4.11)$$

where the quantity $s = 1 - \xi$ is called the slipping of the rotor. Comparing the expressions (4.10) with (4.11) we see that the electromechanical moment of an asynchronous machine with the squirrel-cage type rotor and the stator windings with sinusoidally distributed turns, differs from the moment of a similar machine with the uniformly distributed turns.

Figure 4 also shows the curve $T = T(\xi)$ depicting the dependence of the loading moment at the shaft of an asynchronous machine on the angular velocity of rotation of the rotor.

The character of intersection of the curves $M_* = M_*(\xi)$ and $T = T(\xi)$ is determined by the relation between the parameters. Two of the possible cases are depicted in Fig. 4. In Fig. 4a we have the unique, stable, stready-state mode. In Fig. 4b we have an unstable steady-state mode when the angular velocity $\xi = \xi_1$, and a stable mode when $\xi = \xi_2$. The value $\xi = \xi_1$ represents a limit for the initial values of the angular velocity; when $\xi < \xi_1$ the rotor returns to the state of rest, while for $\xi > \xi_1$, it assumes the state of stable rotation with the angular velocity of $\xi = \xi_2$.

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SHAPING OF A TUBULAR BEAM OF CHARGED PARTICLES EMITTED WITH NONZERO VELOCITY AND IN A NONZERO FIELD STRENGTH

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Within the framework of the hydrodynamic theory of dense beams of charged particles the problem of shaping of a tubular cylindrical flow has been solved. For emission limited by space charge and temperature, shaping electrodes have been constructed; they were computed for exact solution and for asymptotic expansion, equally valid near the flow boundary. The possibility of generalizing the proposed algorithms for the case of curvilinear trajectories is discussed.

In the hydrodynamic theory of dense beams the model of an emitter in the full space charge mode is most widely used. In this case for velocity U and field E, the zero values are taken. These assumptions lead to a fully defined form of the potential φ near the starting surface: in the flow domain $\varphi \sim z^{4/3}$ and in the Laplace domain $\varphi \sim \operatorname{Re}[z + i(R-R_0)]^{4/3}$. As the result we obtain a system of Pierce electrodes with the zero equipotential inclined to the beam boundary at the characteristic initial angle of 67.5°, irrespective of the emitter curvilinearity and of the density of current J at it.

However, in recent times the interest has increased for triode guns with grid control [1 – 4], guns giving sharp deceleration of the flow [5] and guns with auto-emissive cathodes. These structures are distinguished by a more complicated singularity for the potential in the low velocity domain. This singularity for a unidimensional flow between parallel planes are given by the following parametric equations:

$$z = {}^1/_6\, Jt^3 + {}^1/_2\, Et^2 + Ut, \quad v_z = dz / dt, \quad 2\varphi = v_z^2$$

The specific charge $\eta = e / m$ is omitted for reasons of convenience; the change $\eta\varphi \to \varphi$, $4\pi\eta J \to J$, $z = 0$, $t = 0$ correspond to the emitter.

For the structures mentioned above it is necessary to compute the shaping electrodes for the domain where the terms proportional to J, E and U are commensurable. In the triode gun, such a situation occurs when the potential of the control grid deviates from its inherent value, i.e. from the value defined by the ${}^3/_2$ power law. In this case the grid can be considered as an emitter with nonzero conditions on it.